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Relativistic Nonlocal Quantum Field Theory

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A theory is defined to be relativistic if its Hamiltonian, total momenta, and boost's generators satisfy commutation relations of the Poincaré group. Field theories with usual local interactions are known to be relativistic. A simple example of a relativistic nonlocal theory is found. However, it has divergences. Some conditions are obtained which are necessary in order that a nonlocal theory be relativistic and divergenceless.

KEY WORDS: relativistic quantum fields; Poincaré group; nonlocal interactions; regularization of divergences.

1. INTRODUCTION

After Dirac, let us assume the following definition (Dirac, 1949; Leutwyler and Stern, 1978). A theory is called relativistic if it has generators of translations in time (Hamiltonian H), in space (total linear momentum $\vec{P}$), of space rotations (total angular momentum $\vec{M}$), Lorentz boosts $\vec{K}$, and all these generators satisfy the commutation relations of the Lie algebra of the Poincaré group (CPG), e.g., see Gasiorowicz (1966) and Weinberg (1995, chap. 2.4). In Quantum Field Theory (QFT) the generators are expressed in terms of fields. The examples of relativistic QFT are theories of free fields and theories with local interactions. Here, nonlocal QFT are considered which are required to be relativistic in the Dirac sense. Only this property of nonlocal theories is regarded here, other problems are not touched upon (e.g., relativistic causality); see, e.g., Efimov (1985) and Cornish (1992).

For this purpose we use the model of interacting charged ψ and neutral φ scalar fields, see section 2. The model is a simple but representative example of QFT.

The following approach is accepted. Let there be given a theory of the corresponding free fields with generators H_0, $\vec{P}_0$, $\vec{M}_0$, $\vec{K}_0$ satisfying CPG. The generators are expressed in terms of the Schroedinger fields $\varphi(\vec{x})$, $\psi(\vec{x})$ and their conjugates $\pi(\vec{x})$, $\tau(\vec{x})$ for which the usual commutation relations are postulated, see Eq. (2.2). We look for such nonlocal interaction additions to these generators

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that CPG would be still satisfied. The usual form of the theory is assumed (which Dirac called the "instant form") where interaction terms are added only to H_0 and $\vec{K}_0$: $H_0 \to H = H_0 + V$, $\vec{K}_0 \to \vec{K} = \vec{K}_0 + \vec{N}$. The interactions V and $\vec{N}$ are sought as functions of the Schroedinger fields. We need not to consider any nonlocal interaction Lagrangian corresponding to V and $\vec{N}$ (e.g., cf. the paper by Pauli (1953) which starts with a nonlocal Lagrangian). Note that the conservation of energy and momenta is ensured in view of the corresponding CPG containing H, e.g., $[H, \vec{P}] = 0$, etc.

It will be shown in section 4 that the problem posed above has solutions: A simple example of relativistic nonlocal QFT will be given. However, it has divergencies.

Let us add one more requirement to the nonlocal theory sought for: it must be free of divergencies and have a local limit. Such a theory may be used as one possible way for regularization of the corresponding local theory. Note that this way does not need additional (compensative) fields.

The most akin to this work are papers by Kita (1966, 1968). The main difference is that Kita considered models which have no local limits, e.g., the Lee model.

Some necessary conditions for the existence of a relativistic nonlocal theory are obtained here, see Conclusion. I suppose they are noteworthy enough to justify their detailed derivations in section 4.

Remark that the paper deals only with the algebraic aspects of QFT: only commutation relations for fields are needed to calculate CPG, we do not need to introduce any space of quantum states in which fields would be operators. Here fields are considered as elements of a noncommutative algebra with involution † (which corresponds to the Hermitian adjoint in operator representations). Sometimes it is difficult to avoid using the word "operator," but everywhere it means "algebraic element."

2. TRILINEAR INTERACTIONS OF CHARGED AND NEUTRAL SCALAR FIELDS

The free part $H_0 = \int d^3x\, H_0(\vec{x})$ of the total Hamiltonian of the charged ψ and neutral φ scalar fields has the density (e.g., see Wentzel (1949, sections 6 and 8)

$$H_0(\vec{x}) = \frac{1}{2}[\pi^2(\vec{x}) + \vec{\nabla}\varphi(\vec{x}) \cdot \vec{\nabla}\varphi(\vec{x}) + \mu^2\varphi^2(\vec{x})] + [\tau(\vec{x})\tau^\dagger(\vec{x}) + \vec{\nabla}\psi^\dagger(\vec{x}) \cdot \vec{\nabla}\psi(\vec{x}) + m^2\psi^\dagger(\vec{x})\psi(\vec{x})]. \quad (2.1)$$

The usual commutators of the fields are postulated, e.g., nonzero commutators are

$$[\varphi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y}), \qquad [\psi(\vec{x}), \tau(\vec{y})] = i\delta(\vec{x} - \vec{y}),$$
$$[\psi^\dagger(\vec{x}), \tau^\dagger(\vec{y})] = i\delta(\vec{x} - \vec{y}). \quad (2.2)$$

Here π and τ are conjugated to φ and ψ respectively. Usually, one considers the trilinear local interaction

$$V_l = g \int d^3x \varphi(\vec{x})\psi^\dagger(\vec{x})\psi(\vec{x}). \tag{2.3}$$

Its nonlocal generalization may be taken in the form

$$V = \int d^3x \int d^3y \int d^3z \Psi(\vec{x}, \vec{y}, \vec{z})\varphi(\vec{x})\psi^\dagger(\vec{y})\psi(\vec{z}). \tag{2.4}$$

It will be shown in section 4 that the theory with such V can be relativistic only if it is local: $\Psi(\vec{x}, \vec{y}, \vec{z}) \sim \delta(\vec{x} - \vec{y})\delta(\vec{x} - \vec{z})$. To find nonlocal relativistic theories I shall consider additional interactions. The consideration can be decisively simplified if one takes a restricted class of such interactions. Namely, we require that the following conditions are satisfied: (a) interactions must be Hermitian, i.e., $V = V^\dagger$; (b) they must conserve the total charge Q; they must be invariant under (c) the charge conjugation C; and (d) time and space inversions T and $\dot{I}$.

The requirement (b) means that interactions must commute with Q

$$Q = -i \int d^3x[\tau(\vec{x})\psi(\vec{x}) - \tau^\dagger(\vec{x})\psi^\dagger(\vec{x})],$$

see Wentzel (1949, chap. II, section 8). One can directly verify that Q commutes with the following bilinear combinations of charged fields

$$\psi^\dagger(\vec{y})\psi(\vec{z}), \quad \tau(\vec{y})\tau^\dagger(\vec{z}), \quad \tau(\vec{y})\psi(\vec{z}), \quad \psi^\dagger(\vec{y})\tau^\dagger(\vec{z}), \tag{2.5}$$

or their superpositions like $\int d^3y \int d^3z A(\vec{y}, \vec{z})\psi^\dagger(\vec{y})\psi(\vec{z})$ with any c-number function $A(\vec{y}, \vec{z})$.

The charge conjugation C can be defined by the following equations:

$$\begin{aligned} C\psi C^{-1} &= \eta_c\psi^\dagger; \qquad & C\psi^\dagger C^{-1} &= \eta_c^*\psi; \\ C\tau^\dagger C^{-1} &= \eta_c\tau; & C\tau C^{-1} &= \eta_c^*\tau^\dagger; \\ C\varphi C^{-1} &= \varphi; & C\pi C^{-1} &= \pi; \end{aligned} \tag{2.6}$$

$|\eta_c| = 1$; e.g., see Schweber (1961, Eq. (7.335)).

One can construct from (2.5) three C-invariant combinations: $\psi^\dagger(\vec{y})\psi(\vec{z}) + \psi(\vec{y})\psi^\dagger(\vec{z})$ or

$$\int d^3y \int d^3z \Psi_s(\vec{y}, \vec{z})\psi^\dagger(\vec{y})\psi(\vec{z}) \tag{2.7}$$

(here Ψ_s must be a symmetric function $\Psi_s(\vec{y}, \vec{z}) = \Psi_s(\vec{z}, \vec{y})$),

$$\int d^3y \int d^3z T_s(\vec{y}, \vec{z})\tau(\vec{y})\tau^\dagger(\vec{z}), \qquad T_s(\vec{y}, \vec{z}) = T_s(\vec{z}, \vec{y}), \tag{2.8}$$

$$\int d^3y \int d^3z A(\vec{y}, \vec{z})[\tau(\vec{y})\psi(\vec{z}) + \psi^\dagger(\vec{z})\tau^\dagger(\vec{y})], \tag{2.9}$$

A being any c-number function of $\vec{y}, \vec{z}$.

Multiplying (2.7)–(2.9) by neutral fields φ or π, one may construct trilinear combinations which conserve charge and are C-invariant. However, some of them are not invariant under time inversion T:

$$\begin{aligned} &T\varphi(\vec{x})T^{-1} = \eta_0\varphi(\vec{x}); \qquad && T\pi(\vec{x})T^{-1} = -\eta_0\pi(\vec{x}); \\ &T\psi(\vec{x})T^{-1} = \eta\psi(\vec{x}); && T\psi^\dagger(\vec{x})T^{-1} = \eta^*\psi^\dagger(\vec{x}); \\ &T\tau(\vec{x})T^{-1} = -\eta^*\tau(\vec{x}); && T\tau^\dagger(\vec{x})T^{-1} = -\eta\tau^\dagger(\vec{x}). \end{aligned} \tag{2.10}$$

Here $|\eta_0| = |\eta| = 1$ and T is antilinear so that, e.g., $Ti\varphi T^{-1} = -\eta_0 i\varphi$; see Bjorken and Drell (1965, chap. 15). We let $\eta_0 = 1$ and retain only those trilinear interactions which do not change the sign under T-inversion (omitting, e.g., $\pi\psi^\dagger\psi$).

Finally, we get the following most general trilinear interaction satisfying the above requirements (a)–(d)

$$\begin{aligned} V = \int d^3x \int d^3y \int d^3z \{&\Psi(\vec{x}, \vec{y}, \vec{z})\varphi(\vec{x})\psi^\dagger(\vec{y})\psi(\vec{z}) \\ &+ T(\vec{x}, \vec{y}, \vec{z})\varphi(\vec{x})\tau(\vec{y})\tau^\dagger(\vec{z}) + \Pi(\vec{x}, \vec{y}, \vec{z})\pi(\vec{x})[\tau(\vec{y})\psi(\vec{z}) + \psi^\dagger(\vec{z})\tau^\dagger(\vec{y})]\}. \end{aligned} \tag{2.11}$$

The c-number coefficients Ψ and T must be symmetric in the sense

$$\Psi(\vec{x}, \vec{y}, \vec{z}) = \Psi(\vec{x}, \vec{z}, \vec{y}) \qquad T(\vec{x}, \vec{y}, \vec{z}) = T(\vec{x}, \vec{z}, \vec{y}), \tag{2.12}$$

see (2.7) and (2.8); V is hermitian if Ψ, T, Π are real functions.

I do not dwell on invariance under space rotation. It holds if Ψ, T, Π are even functions, e.g.,

$$\Psi(\vec{x}, \vec{y}, \vec{z}) = \Psi(-\vec{x}, -\vec{z}, -\vec{y}). \tag{2.13}$$

However, the property follows from V invariance under space translations and rotations, see the next section.

The interaction terms $\vec{N}$ entering into the boost generators $\vec{K} = \vec{K}_0 + \vec{N}$, $\vec{K}_0 = \int d^3x\vec{x}H_0(\vec{x})$ (see Introduction) are taken in the form analogous to (2.11)

$$\begin{aligned} N^j = \int d^3x \int d^3y \int d^3z \{&\Psi^j\varphi\psi^\dagger\psi + T^j\varphi\tau\tau^\dagger \\ &+ \Pi^j\pi[\tau\psi + \psi^\dagger\tau^\dagger]\}, \quad j = 1, 2, 3. \end{aligned} \tag{2.14}$$

Here the functions Ψ^j, T^j are symmetric in the sense (2.12).

In the "instant form" the generators of space translations $\vec{P}$ and rotations $\vec{M}$ are the same as in the free theory without interactions and, therefore, the CPG

$$[P_i, P_j] = 0; \qquad [M_i, P_j] = i\epsilon_{ijk}P_k; \qquad [M_i, M_j] = i\epsilon_{ijk}M_k \tag{2.15}$$

hold. In the remaining sections 3, 4, and 5, we consider CPG which include $H = H_0 + V$ and $\vec{K} = \vec{K}_0 + \vec{N}$.

3. POINCARÉ COMMUTATORS LINEARLY DEPENDENT ON H AND $\vec{K}$

Let us consider the commutators of H and $\vec{K}$ with the (free) generators $\vec{P} = \vec{P}_0$ and $\vec{M} = \vec{M}_0$

$$[H, P^i] = 0; \qquad [H, M^i] = 0; \tag{3.1}$$

$$[N^i, P^j] = i\delta_{ij}H; \qquad [M^i, N^j] = i\epsilon_{ijk}N^k, \quad i, j, k = 1, 2, 3. \tag{3.2}$$

Using $H = H_0 + V$ and $\vec{K} = \vec{K}_0 + \vec{N}$ and CPG for free generators

$$[H_0, P^i] = 0; \qquad [H_0, M^i] = 0; \tag{3.3}$$

$$\left[N_0^i, P^j\right] = i\delta_{ij}H_0; \qquad \left[M^i, K_0^j\right] = i\epsilon_{ijk}K_0^k, \tag{3.4}$$

one can rewrite (3.1) and (3.2) in terms of V and $\vec{N}$:

$$[V, P^i] = 0; \qquad [V, M^i] = 0; \tag{3.5}$$

$$[N^i, P^j] = i\delta_{ij}V; \qquad [M^i, N^j] = i\epsilon_{ijk}N^k. \tag{3.6}$$

Eqs. (3.5) mean that V must be invariant under space translations and rotations. Therefore, the functions Ψ, T, Π in Eq. (2.11) must depend only upon differences of their arguments, i.e., upon $\vec{x} - \vec{y} \equiv \vec{r}$, $\vec{x} - \vec{z} \equiv \vec{s}$ and, moreover, upon rotation invariants r^2, s^2, $(\vec{r} \cdot \vec{s})$:

$$\Psi(\vec{x}, \vec{y}, \vec{z}) = \Psi(r^2, s^2, (\vec{r} \cdot \vec{s})), \quad \vec{r} = \vec{x} - \vec{y}, \quad \vec{s} = \vec{x} - \vec{z} \tag{3.7}$$

(analogously for T and Π). As Ψ and T are symmetric under the permutation $\vec{y} \leftrightarrow \vec{z}$, they are symmetric under $r^2 \leftrightarrow s^2$.

The second Eq. (3.6) means that N^1, N^2, N^3 make up a three-vector. It follows that the functions Ψ^i, T^i, Π^i in Eq. (2.14) must be components of three-vectors constructed from their vector arguments $\vec{x}$, $\vec{y}$, $\vec{z}$.

The first Eq. (3.6) needs a more thorough consideration. However, at first let us make a decisive simplification. The purpose of this paper is restricted: we do not strive to find all nonlocal versions of the model under consideration; we merely want to give some examples of such versions. It turns out that the examples still exist if Π in Eq. (2.11) and $\vec{\Pi}$ in Eq. (2.14) are forced to be zero.

Returning to the first Eq. (3.6) we calculate the commutator $[N^i, P^j]$ letting $\vec{\Pi} = 0$. We use the commutators

$$[\varphi(\vec{x}), P^j] = -i\partial\varphi/\partial x_j, \qquad [\psi(\vec{x}), P^j] = -i\partial\psi/\partial x_j$$

and analogous ones for other fields π, $\psi^\dagger$, τ, $\tau^\dagger$ (see Eqs. (2.2) and expressions for P^j given in Wentzel (1949, chap. 2). Carrying in $[N^i, P^j] - i\delta_{ij}V$ derivations from fields to the functions Ψ^i, T^i (integration by parts), we obtain

$$\begin{aligned}[N^i, P^j] - i\delta_{ij}V = i\int d^3x \int d^3y \int d^3z \\ \times \left\{\left[\frac{\partial\Psi^i}{\partial x_j} + \frac{\partial\Psi^i}{\partial y_j} + \frac{\partial\Psi^i}{\partial z_j} - \delta_{ij}\Psi\right]\varphi(\vec{x})\psi^\dagger(\vec{y})\psi(\vec{z})\right. \\ \left. + \left[\frac{\partial T^i}{\partial x_j} + \frac{\partial T^i}{\partial y_j} + \frac{\partial T^i}{\partial z_j} - \delta_{ij}T\right]\varphi(\vec{x})\tau(\vec{y})\tau^\dagger(\vec{z})\right\} = 0. \quad (3.8)\end{aligned}$$

A tedious general solution of Eq. (3.8) will not be presented because in the next section we obtain simple expressions (4.7) and (4.8) following from other CPG. They turn into zero the square brackets in Eq. (3.8) and, therefore, the first equation (3.6) does hold.

4. POINCARÉ COMMUTATORS NONLINEAR IN H AND $\vec{K}$: TRILINEAR TERMS

Let us consider the remaining CPG which are nonlinear in H and $\vec{K}$

$$[K^j, H] = iP^j, \qquad [K^i, K^j] = -i\epsilon_{ijk}M^k, \quad i, j, k = 1, 2, 3; \qquad (4.1)$$

Using (4.1) and CPG for free parts H_0 and $\vec{K}_0$ of H and $\vec{K}$ one obtaines the following equations for V and $\vec{N}$

$$[N^j, H_0] + \left[K_0^i, V\right] + [N^j, V] = 0; \qquad (4.2)$$

$$\left[K_0^i, N^j\right] + \left[N^i, K_0^j\right] + [N^i, N^j] = 0. \qquad (4.3)$$

Remember that V and $\vec{N}$ are supposed to be of the trilinear form given by Eqs. (2.11) and (2.14). Then, the first two terms in the l.h.s. of Eqs. (4.2) and (4.3) are also trilinear, while the last ones are quadrilinear. Therefore, the sum of the first two and the latter must vanish separately.

Indeed, consider multiple commutators of the fields with the l.h.s. of (4.2) of (4.3) of the kind that will be used below (1), but fourfold ones, e.g., $[\pi, [\pi, [\psi, [\tau, (4.2)]]]]$. They turn into zero the first two terms of (4.2) and (4.3) and turn the last ones into c-number. As the l.h.s. of (4.2) and (4.3) are zero, so are these c-numbers and, therefore, the last terms must vanish. Then, the sum of the first two must also vanish separately.

1. Let us seek for V and $\vec{N}$ which would cancel trilinear terms in l.h.s. of Eq. (4.2)

$$\left[K_0^j, V\right] - [H_0, N^j] = \int d^3x \{x_j [H_0(\vec{x}), V] - [H_0(\vec{x}), N^j]\}. \quad (4.4)$$

This would mean the finding of some necessary conditions for the fulfilment of (4.1). They may be insufficient ones because the quadrilinear terms in (4.2) also must vanish.

The calculation of the commutators in (4.4) are straightforward though tedious. The encounting commutators of the kind

$$\left[\frac{\partial}{\partial x_j}\varphi(\vec{x}), \pi(\vec{x})\right] = i\partial\delta(\vec{x} - \vec{x}')/\partial x_j$$

follow from (2.2). Integrations by parts are used, e.g.,

$$\int d^3x' \sum_{i=1}^{3} \frac{\partial}{\partial x_i'}\varphi(\vec{x}')\frac{\partial}{\partial x_i'}\delta(\vec{x}' - \vec{x}) = -\Delta_x\varphi(\vec{x}),$$

$$\Delta_x \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2 \quad (4.5)$$

as well as changes of variables which numerate (are arguments of) fields. The result is

$$\begin{aligned}
&\left[K_0^j, V\right] - [H_0, N^j] \\
&= -i\int d^3x \int d^3y \int d^3z \{[x_j\Psi - \Psi^j]\pi(\vec{x})\dot{\psi}^\dagger(\vec{y})\psi(\vec{z}) \\
&+ [y_j\Psi - \Psi^j + \partial T/\partial z_j + z_j(\Delta_z - m^2)T - (\Delta_z - m^2)T^j] \\
&\times \varphi(\vec{x})(\tau(\vec{y})\psi(\vec{z}) + \psi^\dagger(\vec{z})\tau^\dagger(\vec{y})) \\
&+ [x_jT - T^j]\pi(\vec{x})\tau(\vec{y})\tau^\dagger(\vec{z})\}. \quad (4.6)
\end{aligned}$$

It is evident that the r.h.s. of Eq. (4.6) vanishes if c-number multiples of $\pi\psi^\dagger\psi$, $\varphi(\tau\psi + \psi^\dagger\tau^\dagger)$ and $\pi\tau\tau^\dagger$ vanish. Let us show that the inverse is also true: if (4.6) vanishes then the square brackets in (4.6) vanish separately. Indeed, the threefold commutator

$$[\tau(\vec{z}'), [\tau^\dagger(\vec{y}'), [\varphi(\vec{x}'), r.h.s.(4.6)]]]$$

is equal to multiple of $\pi\psi^\dagger\psi$ in (4.6), i.e. to the first square bracket in (4.6) (it is symmetric under $\vec{y} \leftrightarrow \vec{z}$). Analogously $[\psi^\dagger, [\psi, [\pi, (4.6)]]]$ is equal to the multiple of $\pi\tau\tau^\dagger$. Unlike these commutators the multiple commutator $[\tau(\vec{z}'), [\psi(\vec{y}'), [\pi(\vec{x}'), (4.6)]]]$ is equal to the second square bracket written in (4.6) whereas $[\psi(\vec{z}'), [\tau(\vec{y}'), [\pi(\vec{x}'), (4.6)]]]$ is equal to this bracket with interchanged $\vec{y}$ and $\vec{z}$.

So using algebraic tools only, we get from Eq. (4.2) the following equations for the c-number functions Ψ, Ψ^j, T, T^j:

$$x_j \Psi(\vec{x}, \vec{y}, \vec{z}) - \Psi^j(\vec{x}, \vec{y}, \vec{z}) = 0, \tag{4.7}$$

$$x_j T(\vec{x}, \vec{y}, \vec{z}) - T^j(\vec{x}, \vec{y}, \vec{z}) = 0, \tag{4.8}$$

$$y_j \Psi - \Psi^j + z_j(\Delta_z - m^2)T - (\Delta_z - m^2)T^j = 0, \tag{4.9}$$

$$z_j \Psi - \Psi^j + y_j(\Delta_y - m^2)T - (\Delta_y - m^2)T^j = 0, \tag{4.10}$$

These equations must hold for all $\vec{x}, \vec{y}, \vec{z}$, and $j = 1, 2, 3$.

2. In an analogous manner one can obtain equations for Ψ, Ψ^j, T, T^j resulting from vanishing of the trilinear terms in Eq. (4.3). One can show that the terms vanish if Eqs. (4.7)–(4.10) hold. So in what follows we may consider the latter ones only.
3. Substituting the solutions $\Psi^j = x_j \Psi$ and $T^j = x_j T$ of Eqs. (4.7) and (4.8) into Eqs. (4.9) and (4.10), one obtains

$$-r_j \Psi - \partial T/\partial z_j - s_j(\Delta_z - m^2)T = 0, \tag{4.11}$$

$$-s_j \Psi - \partial T/\partial y_j - r_j(\Delta_y - m^2)T = 0. \tag{4.12}$$

Here $r_j \equiv x_j - y_j$, $s_j \equiv x_j - z_j$. Remind that Ψ and T are functions of $\vec{r}$ and $\vec{s}$, see section 3, and, therefore,

$$\partial T/\partial z_j = -\partial T/\partial s_j; \qquad \partial T/\partial y_j = -\partial T/\partial r_j;$$

$$\Delta_z T = \Delta_s T; \qquad \Delta_y T = \Delta_r T.$$

Here Δ is Laplacian, see Eq. (4.5).

Eqs. (4.11) and (4.12) are partial derivative equations of the second order. In the momentum representation they turn into simpler equations of the first order. To obtain the latter, multiply the l.h.s. of Eqs. (4.11) and (4.12) by $\exp i(\vec{p}\cdot\vec{r} + \vec{q}\cdot\vec{s})$ and integrate over $\vec{r}, \vec{s}$. Denoting

$$\tilde{\Psi}(\vec{p}, \vec{q}) = \int d^3r \int d^3s \Psi(\vec{r}, \vec{s}) \ \exp i(\vec{p}\cdot\vec{r} + \vec{q}\cdot\vec{s}) \tag{4.13}$$

and using the equation of the kind

$$\int d^3r \int d^3s s_j(\Delta_s - m^2)T(\vec{r}, \vec{s}) \ \exp i(\vec{r}\cdot\vec{r} + \vec{q}\cdot\vec{s})$$
$$= -i\frac{\partial}{\partial q_j}[-q^2 - m^2]\tilde{T}(\vec{p}, \vec{q}) = i[2q_j\tilde{T} + (q^2 + m^2)\partial T/\partial q_j], \tag{4.14}$$

one obtains

$$\partial\tilde{\Psi}/\partial p_j - q_j\tilde{T} - (q^2 + m^2)\partial\tilde{T}/\partial q_j = 0, \quad \forall\vec{p}, \vec{q}, \; j = 1, 2, 3; \quad (4.15)$$

$$\partial\tilde{\Psi}/\partial q_j - p_j\tilde{T} - (p^2 + m^2)\partial\tilde{T}/\partial p_j = 0. \quad (4.16)$$

It follows from Eq. (3.7) that $\tilde{\Psi}(\vec{p}, \vec{q}) = \Psi(p^2, q^2, \vec{p}\cdot\vec{q})$. Let us use arguments $\epsilon_p = \sqrt{p^2 + m^2}$ and $\epsilon_q = \sqrt{q^2 + m^2}$ instead of p^2 and q^2 and denote $t = \vec{p}\cdot\vec{q}$. Then, Eqs. (4.15) turn into

$$\vec{p}\left(\frac{1}{\epsilon_p}\partial\tilde{\Psi}/\partial\epsilon_p - \epsilon_q^2\partial\tilde{T}/\partial t\right) + \vec{q}(\partial\tilde{\Psi}/\partial t - \tilde{T} + \epsilon_p\partial\tilde{T}/\partial\epsilon_q) = 0. \quad (4.17)$$

Eqs. (4.16) turn into equations (4.17t) which are Eqs. (4.17) with transposed $\vec{p}$ and $\vec{q}$.

From Eqs. (4.17) we obtain two equations

$$\frac{1}{\epsilon_p}\partial\tilde{\Psi}/\partial\epsilon_p - \epsilon_q^2\partial\tilde{T}/\partial t = 0; \quad \partial\tilde{\Psi}/\partial t - \tilde{T} - \epsilon_q\partial\tilde{T}/\partial\epsilon_q = 0, \quad (4.18)$$

using vector (outer) products at first by $\vec{p}$ and then by $\vec{q}$ ($\vec{p}$ is supposed to be not parallel to $\vec{q}$ so that $\vec{p}\times\vec{q} \neq 0$).

From transposed Eq. (4.17t) one obtains Eq. (4.18) where ϵ_p and ϵ_q are transposed

$$\frac{1}{\epsilon_q}\partial\tilde{\Psi}/\partial\epsilon_q - \epsilon_p^2\partial\tilde{T}/\partial t = 0; \quad \partial\tilde{\Psi}/\partial t - \tilde{T} - \epsilon_p\partial\tilde{T}/\partial\epsilon_p = 0. \quad (4.19)$$

So we have reduced the starting Eqs. (4.7)–(4.10) to the system (4.18) and (4.19) of partial derivative equations of the first order for the c-number functions. Their general solution is obtained in Appendix A:

$$\tilde{\Psi}(\vec{p}, \vec{q}) = -f_1(\epsilon_p\epsilon_q - \vec{p}\cdot\vec{q}) + f_2(\epsilon_p\epsilon_q + \vec{p}\cdot\vec{q}), \quad (4.20)$$

$$\tilde{T}(\vec{p}, \vec{q}) = \frac{1}{\epsilon_p\epsilon_q}[f_1(\epsilon_p\epsilon_q - \vec{p}\cdot\vec{q}) + f_2(\epsilon_p\epsilon_q + \vec{p}\cdot\vec{q})]. \quad (4.21)$$

Here f_1 and f_2 are arbitrary functions of their arguments. Let us discuss some particular cases of the solution.

4. Let the interaction V be given by Eq. (2.4), i.e., $\tilde{\Psi} \neq 0$, $\tilde{T} = 0$. The solution (4.21) can be zero at all ϵ_p, ϵ_q, $(\vec{p}\cdot\vec{q})$ if

$$f_1(\omega - t) + f_2(\omega + t) = 0, \quad \omega \equiv \epsilon_p\cdot\epsilon_q, \quad t \equiv \vec{p}\cdot\vec{q} \quad (4.22)$$

for all values of ω, t. This is a functional equation. Consider the set of ω, t values which satisfy $\omega - t = \text{constant}$. For such ω, t values f_1 in (4.11) is a constant C, while the argument $\omega + t$ of the function f_2 does vary: However, (4.22) states that f_2 is nevertheless equal to the constant $-C$. Then, f_1 is equal to $+C$ at all ω, t. So $\tilde{\Psi} = -f_1 + f_2 = -2C$ is constant and its Fourier prototype Ψ (see Eq. (4.13)) is proportional to the product $\delta(\vec{x} - \vec{y})\delta(\vec{x} - \vec{z})$

$$\Psi(\vec{r}, \vec{s}) \sim \int d^3p \int d^3q 2C \ \exp(-i)(\vec{p} \cdot \vec{r} + \vec{q} \cdot \vec{s}) \sim \delta(\vec{r})\delta(\vec{s}). \quad (4.23)$$

This result may be formulated as a “no-go theorem”: The relativistic nonlocal theory does not exist if the interaction is of the kind $\int \int \int \Psi \varphi \psi^\dagger \psi$ only, i.e., there is no admixture of other interactions.

The conclusion can be derived immediately from Eqs. (4.7)–(4.10): if $T = 0$, then Eqs. (4.9) and (4.10) turn into equations $(y_j - x_j)\Psi = 0$ and $(z_j - x_j)\Psi = 0$. Their nonzero solution is $\Psi(\vec{x}, \vec{y}, \vec{z}) \sim \delta(\vec{x} - \vec{y})$ $\delta(\vec{x} - \vec{z})$.

Let us stress that in this particular case not only the trilinear parts of (4.2) and (4.3) vanish but also the remaining quadrilinear ones because

$$[\varphi(\vec{x})\psi^\dagger(\vec{y})\psi(\vec{z}), \varphi(\vec{x}')\psi^\dagger(\vec{y}')\psi(\vec{z}')] = 0.$$

So the obtained particular solution $\tilde{T} = 0$, $\tilde{\Psi} =$ constant turns out to be not only a necessary condition, but also a sufficient one in order that the obtained local theory be relativistic.

5. Let $\tilde{\Psi} = 0$, $\tilde{T} \neq 0$. The restriction $\tilde{\Psi} = -f_1 + f_2 = 0$ leads to $f_1 = f_2 = C$. Then $\tilde{T} = 2C/\epsilon_p \epsilon_q$. This solution can be obtained more simply from Eqs. (4.18) and (4.19) where $\tilde{\Psi}$ is put equal to zero. The corresponding Fourier prototype $T(\vec{x}, \vec{y}, \vec{z})$ is not local. The interactions are

$$V_T = \int \int \int T \varphi \tau \tau^\dagger; \qquad N_T^j = \int \int \int T^j \varphi \tau \tau^\dagger; \quad T^j = x_j T. \tag{4.24}$$

As in the previous case, the quadrilinear terms in Eqs. (4.2) and (4.3) vanish. So we get a simple example of the relativistic QFT which is nonlocal. Moreover, the relativistic local theory does not exist if interaction terms are of the kind (4.24).

Let us show that in this case the theory has the same divergencies as in the previous local case. For this purpose, use the well-known expansions of $\varphi, \pi; \psi, \tau^\dagger; \psi^\dagger, \tau$ in the creation–destruction operators g, $g^\dagger; a, b^\dagger; a^\dagger, b$, respectively (e.g., see Wentzel (1949, chap. 2). Then, the interaction V, given by Eq. (2.11) with $\Pi = 0$ can be represented as follows:

$$\begin{aligned} V &= (\pi)^{\frac{3}{2}} \int d^3k \int d^3p \int d^3q \delta(\vec{k} - \vec{p} + \vec{q})(\omega_k \epsilon_p \epsilon_q)^{-\frac{1}{2}} \\ &\quad \times (g_{\vec{k}} + g^{\dagger}_{-\vec{k}})[V_{11}(\vec{p}, \vec{q}) a^{\dagger}_{\vec{p}} a_{\vec{q}} + V_{12}(\vec{p}, \vec{q}) a^{\dagger}_{\vec{p}} b^{\dagger}_{-\vec{q}} \\ &\quad + V_{21}(\vec{p}, \vec{q}) b_{-\vec{p}} a_{\vec{q}} + V_{22}(\vec{p}, \vec{q}) b_{-\vec{p}} b^{\dagger}_{-\vec{q}}], \end{aligned} \tag{4.25}$$

$$\begin{aligned} V_{11}(\vec{p}, \vec{q}) &= V_{22}(\vec{p}, \vec{q}) = \tilde{\Psi}(\vec{p}, -\vec{q}) + \epsilon_p \epsilon_q \tilde{T}(\vec{p}, -\vec{q}) \\ &= 2 f_2(\epsilon_p \epsilon_q - \vec{p} \cdot \vec{q}), \\ V_{12}(\vec{p}, \vec{q}) &= V_{21}(\vec{p}, \vec{q}) = \tilde{\Psi}(\vec{p}, -\vec{q}) - \epsilon_p \epsilon_q \tilde{T}(\vec{p}, -\vec{q}) \\ &= -2 f_1(\epsilon_p \epsilon_q + \vec{p} \cdot \vec{q}). \end{aligned} \tag{4.26}$$

Equation (4.13) has been used. We see that in both the cases $\tilde{\Psi} =$ constant, and $\tilde{T} = 0$ and $\tilde{\Psi} = 0$, $\tilde{T} = \text{constant}/\epsilon_p \epsilon_q$ all coefficients V_{mn} are constants.

Now let us consider the cases when f_1 and f_2 are not constants.

6. We see immediately that the allowed solution $\tilde{\Psi}(\vec{p}, \vec{q})$ cannot depend on $\vec{p}$ only (or on $\vec{q}$ only): it must depend on both $\vec{p}$ and $\vec{q}$ by means of the combinations $\epsilon_p \epsilon_q \pm \vec{p} \cdot \vec{q}$. So does $\tilde{T}$. This means that theories with nonlocal interactions of the kind

$$\int\int \Psi(\vec{x} - \vec{z}) \varphi(\vec{x}) \psi^{\dagger}(\vec{x}) \psi(\vec{z}) + \int\int \tilde{T}(\vec{x} - \vec{z}) \varphi(\vec{x}) \tau(\vec{x}) \tau^{\dagger}(\vec{z}) \tag{4.27}$$

cannot be relativistic.

7. It is possible to choose such particular solutions f_1 and f_2 that divergencies will be suppressed and the local interaction (2.3) will emerge in a limit. Indeed, put for example

$$\begin{aligned} f_1 &= -C \ \exp[-(\epsilon_p \epsilon_q - \vec{p} \cdot \vec{q})/M^2], \\ f_2 &= C \ \exp[-(\epsilon_p \epsilon_q + \vec{p} \cdot \vec{q})/M^2]. \end{aligned} \tag{4.28}$$

Here M denotes a cutoff parameter. Then

$$\tilde{\Psi} = 2C \ \exp(-\epsilon_p \epsilon_q / M^2) \ \cosh(\vec{p} \cdot \vec{q})/M^2, \tag{4.29}$$

$$\epsilon_p \epsilon_q \tilde{T} = -2C \ \exp(-\epsilon_p \epsilon_q / M^2) \ \sinh(\vec{p} \cdot \vec{q})/M^2. \tag{4.30}$$

When $|\vec{p}| \to \infty$ or $|\vec{q}| \to \infty$ we have a cutoff which is able to eliminate any of the known divergencies. In the limit $M \to \infty$, $\tilde{\Psi}$ tends to a constant while $\tilde{T}$ vanishes, i.e., the interaction $\int \int \int \Psi \varphi \psi^{\dagger} \psi$ tends to the local one (2.3) while $\int \int \int T \varphi \tau \tau^{\dagger}$ tends to zero.

However, the problem of canceling the quadrilinear parts of (4.2) and (4.3) arises in this example. This will be outlined in the next section.

5. POINCARÉ COMMUTATORS NONLINEAR IN H AND $\vec{K}$ BEYOND TRILINEAR TERMS

Let us calculate the commutator $[N^j, V]$ entering into Eq. (4.2) using V and N^j found in the previous section. We obtain

$$[N^j, V] = -i \int d^3x \int d^3x' \int d^3y \int d^3z F_j(\vec{x}, \vec{x}', \vec{y}, \vec{z})\varphi(\vec{x})\varphi(\vec{x}') \times [\tau(\vec{y})\psi(\vec{z}) + \psi^\dagger(\vec{z})\tau^\dagger(\vec{y})], \tag{5.1}$$

$$F_j(\vec{x}, \vec{x}', \vec{y}, \vec{z}) = \int d^3u(x'_j - x_j)[\Psi(\vec{x}, \vec{u}, \vec{z})T(\vec{x}', \vec{y}, \vec{u}) - \Psi(\vec{x}', \vec{u}, \vec{z})T(\vec{x}, \vec{y}, \vec{u})]. \tag{5.2}$$

The commutator $[N^j, V]$ vanishes if $F_j = 0$; F_j does vanish evidently in the particular cases when either T or Ψ are zero. It can be shown that there are no other cases when $F_j = 0$. Taking the omitted terms with Π and $\vec{\Pi}$ into account does not seem to help the trouble of nonvanishing $[N^j, V]$ in any way. Following Kita (1966, 1968), one may suggest the following schematic approach to provide the fulfilment of the commutators (4.1).

Let trilinear interactions V and N^j be proportional to a coupling constant g. In what follows denote them by gV_3 and gN_3^j. Let us add quadrilinear interactions $\sim g^2$ so that

$$V = gV_3 + g^2V_4, \qquad N^j = gN_3^j + g^2N_4^j. \tag{5.3}$$

Then, we obtain

$$\begin{aligned}[K^j, H] - iP^j = g\{[K_0^j, V_3] + [N_3^j, H_0]\} &+ g^2\{[K_0^j, V_4] + [N_4^j, H_0] + [N_3^j, V_3]\} \\ &+ g^3\{[N_3^j, V_4] + [N_4^j, V_3]\} + g^4[N_4^j, V_4]\end{aligned} \tag{5.4}$$

(analogously for the second commutator in (4.1)). The interactions g^2V_4 and $g^2N_4^j$ are to cancel the terms $\sim g^2$ in (5.4). For this purpose, the commutators $[K_0^j, V_4] + [N_4^j, H_0]$ in the r.h.s. of Eq. (5.4) must contain the terms $\varphi\varphi(\tau\psi + \psi^\dagger\tau^\dagger)$, see Eq. (5.1). To provide this, V_4 and N_4^j must contain

$$\varphi\varphi\psi^\dagger\psi, \quad \varphi\varphi\tau\tau^\dagger, \quad \pi\varphi(\tau\psi^\dagger + \psi^\dagger\tau^\dagger). \tag{5.5}$$

But then some excess undesired terms of the kind $\pi\varphi\psi^\dagger\psi$, $\pi\varphi\tau\tau^\dagger$ will appear in $[K_0^j, V_4] + [N_4^j, H_0]$. For their compensation one should add to (5.5) the terms $\pi\pi\psi^\dagger\psi$ and $\pi\pi\tau\tau^\dagger$. As the result, all the terms $\sim g^2$ in the r.h.s. of Eq. (5.4) must vanish. The terms of the order g^3 and g^4 also must vanish. For their canceling one ought to add in the r.h.s. of Eq. (5.3) pentalinear, hexalinear, etc., interactions.

Let us note one consequence of this approach. Suppose one wants to calculate an effect of the order g^2 using the described nonlocal theory. Then, one must take into account not only the trilinear interaction gV_3 but also the interaction g^2V_4 constructed above.

6. CONCLUSION

The commutation relations of the Poincaré group have been considered for interacting neutral φ and charged ψ fields. It has been proved that if their interactions are of the kind $\int\int\int \Psi(\vec{x}, \vec{y}, \vec{z})\varphi(\vec{x})\psi^\dagger(\vec{y})\psi(\vec{z})$, then the theory can be relativistic in the Dirac sense in the case of the local interaction only: $\Psi(\vec{x}, \vec{y}, \vec{z}) \sim \delta(\vec{x} - \vec{y})\delta(\vec{x} - \vec{z})$.

If the interaction is of the kind $\int\int\int T(\vec{x}, \vec{y}, \vec{z})\varphi(\vec{x})\tau(\vec{y})\tau^\dagger(\vec{z})$, ($\tau$ being the conjugate to ψ), the theory is shown to be relativistic only if T is a nonlocal function of $\vec{x}, \vec{y}, \vec{z}$. However, this case has the same divergencies as the previous local one.

Some necessary conditions for the existence of the relativistic nonlocal theory without divergencies has been obtained. First of all, interactions must be superpositions of terms including, in particular, $\Psi\varphi\psi^\dagger\psi$ and $T\varphi\tau\tau^\dagger$. The obtained conditions do not specify the explicit forms of the corresponding formfactors $\tilde{\Psi}(\vec{p}, \vec{q})$ and $\tilde{T}(\vec{p}, \vec{q})$ or $f_1(\vec{p}, \vec{q})$ and $f_2(\vec{p}, \vec{q})$, see Eqs. (4.13), (4.20), and (4.21). However, $f_1(\vec{p}, \vec{q})$ and $f_2(\vec{p}, \vec{q})$ are to depend only on specific combinations of $\vec{p}, \vec{q}$, namely on $I_\mp(\vec{p}, \vec{q}) = \sqrt{p^2 + m^2}\sqrt{q^2 + m^2} \mp (\vec{p} \cdot \vec{q})$. The combinations are relativistic invariants constructed from four-vectors $(\sqrt{p^2 + m^2}, \vec{p})$ and $(\sqrt{q^2 + m^2}, \vec{q})$ or $(\sqrt{q^2 + m^2}, -\vec{q})$.

A specific example of cutting off formfactor is given [see subsect 7 in section 4]. Its particular property is that the corresponding nonlocal theory turns into the usual local one (with $\tilde{\Psi} = \text{const}$, $\tilde{T} = 0$) when a cutoff parameter tends to infinity. This example shows that this nonlocal theory can be used as a way of relativistic regularization of the local one.

APPENDIX A: GENERAL SOLUTION OF EQS. (4.18) AND (4.19)

Let us consider the pair of second equations in Eqs. (4.18) and (4.19). Their difference gives

$$\epsilon_q \partial \tilde{T}/\partial \epsilon_q - \epsilon_p \partial \tilde{T}/\partial \epsilon_p = 0. \tag{A1}$$

The corresponding system of ordinary differential equations (Forsyth, 1959) is $d\epsilon_q/\epsilon_q = -d\epsilon_p/\epsilon_p$. It has the integral $\epsilon_p\epsilon_q$ = constant. So the solution $\tilde{T}(\epsilon_p, \epsilon_q, t)$ of Eq. (A1) is an arbitrary function of the variables $\epsilon_p \cdot \epsilon_q$ and t.

From the pair of first equations in Eqs. (4.18) and (4.19) one obtains

$$\epsilon_p \partial\tilde{\Psi}/\partial\epsilon_p - \epsilon_q \partial\tilde{\Psi}/\partial\epsilon_q = 0, \tag{A2}$$

i.e., the same equation as (A1). Its general solution is an arbitrary function $\tilde{\Psi}(\epsilon_p \cdot \epsilon_q, t)$ of $\epsilon_p \cdot \epsilon_q$ and t.

Let us denote $\omega \equiv \epsilon_p\epsilon_q$ and substitute $\tilde{T}(\omega, t)$ and $\tilde{\Psi}(\omega, t)$ in the Eqs. (4.18) and (4.19). Then, Eqs. (4.19) turn out to coincide with Eqs. (4.18) and we get a system of two equations for $\tilde{\Psi}(\omega, t)$ and $\tilde{T}(\omega, t)$:

$$\partial\tilde{\Psi}/\partial\omega - \omega\partial\tilde{T}/\partial t = 0, \qquad \partial\tilde{\Psi}/\partial t - T - \omega\partial\tilde{T}/\partial\omega = 0. \tag{A3}$$

Differentiate the first equation of the system over t and the second one over ω. The difference of the resulting equations turns out to be the equation for $\tilde{T}$ only, but of the second order

$$\omega\partial^2\tilde{T}/\partial t^2 = 2\partial\tilde{T}/\partial\omega + \omega\partial^2\tilde{T}/\partial\omega^2. \tag{A4}$$

Without loss of generality let us introduce a new unknown function f instead of $\tilde{T} : \tilde{T} = \omega^{-1} f$. Then, Eq. (A4) turns into $\partial^2 f/\partial t^2 - \partial^2 f/\omega^2 = 0$. This simplest hyperbolic equation is known to have the general solution of the form

$$f = f_1(\omega - t) + f_2(\omega + t),$$

where f_1 and f_2 are independent arbitrary functions of their arguments $\omega - t$ and $\omega + t$, respectively. Substituting

$$\tilde{T}(\epsilon_p, \epsilon_q, t) = [f_1(\omega - t) + f_2(\omega + t)]/\omega, \quad \omega = \epsilon_p\epsilon_q \tag{A5}$$

into Eqs. (A3), we get

$$\partial\tilde{\Psi}/\partial\omega + f_1' - f_2' = 0, \qquad \partial\tilde{\Psi}/\partial t - (f_1' + f_2') = 0. \tag{A6}$$

Here $f_{1,2}'(x) \equiv df_{1,2}/dx$. Each of these two equations is an ordinary differential equation. In the first one, t may be considered as a parameter and its general solution is

$$\begin{aligned}\tilde{\Psi}(\omega, t) &= \int^{\omega} d\omega'[-f_1'(\omega' - t) + f_2'(\omega' + t)] + C(t)\\ &= \int^{\omega} d\omega'[-\partial f_1/\omega' + \partial f_2/\omega'] + C(t)\\ &= -f_1(\omega - t) + f_2(\omega + t) + C(t),\end{aligned} \tag{A7}$$

where $C(t)$ is an arbitrary function of t. Substituting (A7) into the second equation (A6) one obtains $dC(t)/dt = 0$, i.e., $C(t)$ does not depend on t, and is an arbitrary constant. So

$$\tilde{\Psi}(\epsilon_p, \epsilon_q, t) = -f_1(\epsilon_p \epsilon_q - t) + f_2(\epsilon_p \epsilon_q + t) + C. \tag{A8}$$

The constant C may be included into arbitrary functions f_1 and f_2, and therefore C is omitted in Eqs. (4.20) and (4.21).

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